VIBRATIONS OF A NONHOMOGENEOUS MEDIUM-CONTACTING CYLINDRICAL SHELL STIFFENED WITH RINGS AND SUBJECTED TO ACTION OF COMpressive FORCE

F.S. Latifov  R.A. Iskanderov  K.A. Babayeva

Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
flatifov@mail.ru, r.iskanderov@gmail.com, babyevakonul@lenta.ru

Abstract- The non-homogeneity of a cylindrical shell in thickness may be taken into account by two different methods by introducing sandwich and non-homogeneity function. In this paper the inhomogeneity was taken into account by accepting the Young modulus and the density of the material as a function of coordinate changing in thickness.

Keywords: Three-Dimensional Functional, Nonhomogeneity, Liquid Medium, Variation Principle.

I. INTRODUCTION

Investigation of strength characteristics of medium-contacting constructions and structural elements subjected to the action of external force with regard to nonhomogeneity of distribution of material, influence of medium and compressive force is of great importance.

Parametric vibrations of uniform cylindrical shells with regard to nonhomogeneity in thickness were studied in the papers [1-4]. Using the variational principle in solving the problem, for finding vibration frequencies of the considered system a frequency equation was structured and studied depending on physico-geometrical parameters characterizing the system, the characteristic curves were constructed on the force-frequency plane.

In this papers [5-7], for studying free and forced vibrations of a fluid-filled cylindrical shell stiffened with bars and subjected to the action of compressive force in the axial direction, physical mathematical model was constructed. For the cases of axially symmetric and asymmetric cases of vibrations, the frequency equation of the system was structured and approximate roots were found. The influence of geometrical, physical-mechanical parameters characterizing the system on the found frequencies were studied.

At the same time, the forced vibrations of the system were considered, displacements of the cylinder in resonance frequencies and near it, were calculated. By introducing optimization parameters, the optimal variant of geometrical sizes of the cylindrical shell and the number of bars were found. In the paper, natural vibrations of the system of cylindrical shell, continuum stiffened with non-homogeneous rings and subjected to the action of compressive force are studied.

II. PROBLEM STATEMENT

For taking into account the nonhomogeneity of the cylindrical shell in thickness, we will use a three-dimensional functional. In this case, the total energy of the cylindrical shell is as follows:

\[ U = \frac{1}{2} \int \int \int \left( \sigma_{\alpha} e_{\alpha} + \sigma_{\beta} e_{\beta} + \tau_{\alpha\beta} e_{\alpha\beta} + \rho(z) \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial \varphi}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right) d\alpha d\beta dz \]

(1)

where

\[ \sigma_{\alpha} = \frac{T_{1}}{h} + \frac{12M_{1}}{h^3 - z} \]

(2)

\[ \sigma_{\beta} = \frac{T_{2}}{h} + \frac{12M_{2}}{h^3 - z} \]

\[ \tau_{\alpha\beta} = \frac{s}{h} + \frac{12H}{h^3 - z} \]

The nonhomogeneity is taken into account by different methods. One of them is to accept the Young modulus and the density of the material as a function of coordinate changing in thickness [1]: \( E = E(z) \), \( \rho = \rho(z) \). We assume that the Poisson ratio is constant.

In this case stress-strain relations are written as follows:

\[ e_{\alpha} = \frac{1}{E(z)} \left( \sigma_{\alpha} - \nu \sigma_{\beta} \right) \]

(3)

\[ e_{\beta} = \frac{1}{E(z)} \left( \sigma_{\beta} - \nu \sigma_{\alpha} \right) \]

\[ e_{\alpha\beta} = \frac{2(1+\nu)}{E(h)} \sigma_{\alpha\beta} \]
Taking into account relations (2) and (3), and equality
\[
\int_{-h/2}^{h/2} \left( \rho(z) \left( \frac{\partial w}{\partial t} \right)^2 + \frac{\partial^2 w}{\partial t^2} \right) \left( \frac{\partial w}{\partial t} \right)^2 + \frac{\partial^2 w}{\partial t^2} \right) \right) \, d\alpha \, d\beta \, dz =
\]
\[
= \int \left( \rho_0 \left( \frac{\partial w}{\partial t} \right)^2 \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{\partial^2 w}{\partial t^2} \right) + \rho_2 \left( \frac{\partial^2 \phi}{\partial t^2} \right)^2 \right) \, d\alpha \, d\beta \right)
\]
in Equation (1), we can write:
\[
V = \frac{1}{h} \int T \left[ \frac{12}{E_i h^3} \left( M_z - v M_1 \right) \right] +
\]
\[
+ T_2 \left[ \frac{12}{E_i h^3} \left( M_1 - v M_z \right) \right] +
\]
\[
+ 2(1 + v) s \left( \frac{12 H}{E_i h^3} + \frac{72}{E_i h^5} \times
\]
\[
\times \left[ 2 M_i M_z - v M_i^2 + v M_z^2 + 2(1 + v) H^2 \right] \right) \, d\alpha \, d\beta
\]
Let us write the total energy of the rings:
\[
V_i = \frac{1}{h} \sum_{j=1}^{k_1} \int \left[ E_{i,j} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{w_j}{R} \right]
\]
\[
+ E_{i,j} \frac{\partial^2 w_j}{\partial t \partial y} \left( \frac{\partial^2 u_i}{\partial y^2} + \frac{w_j}{R^2} \right)^2 +
\]
\[
+ G_{i,j} \frac{\partial^2 \phi_{k_{j+1}}}{\partial y^2} \left( \frac{w_i}{R} \right)^2 \right) \, dy
\]
(5)
\[
\sum_{j=1}^{k_2} \rho_j \int \left[ \frac{\partial u_j}{\partial t} \right] \left( \frac{\partial \phi}{\partial t} \right)^2 +
\]
\[
\frac{\partial \phi}{\partial t} \right) \right) \, dy
\]
\[
\sum_{j=1}^{k_2} \rho_j \int \left[ \frac{\partial^2 \phi}{\partial t^2} \right] \right) \, dy
\]
\[
\frac{\partial^2 \phi}{\partial t \partial y} \left( \frac{\partial \phi}{\partial t} \right) \right) \, dy
\]
\[
The influence of medium on the cylindrical shell is replaced by external forces \( q_i, q_j, q_k \). The work done by these forces in displacements of points of the shell is
\[
A_0 = \int_{0}^{2\pi} \left( q_i u + q_j \theta + q_k w \right) \, r \, dr \, d\theta
\]
(6)
Let us write potential energy generated in the shell and bar because of influence of compressive stresses \( \sigma_z \):
\[
\Pi = -\frac{\sigma_z}{2} \int \left( \frac{\partial \phi}{\partial z} \right) \, d\xi \, d\theta \left( \sigma_z F_c \sum_{i=1}^{k} \frac{\partial \phi}{\partial \xi} \right)^2 \right) \, d\xi
\]
(7)
The total energy of the considered system will consist of
\[
W = V + V_i + A_0 + \Pi
\]
(8)
In Equations (1)-(8), \( u, \theta, w \) are displacements of the shell, \( u_i, \theta_i, w_i \) are displacements of the points of the bar, \( E \) is modulus of elasticity of the cylindrical shell’s material and the Poisson ratio, respectively, \( R, h \) are radii and thickness of the cylindrical shell, \( E_i \) is the modulus of elasticity of longitudinal bar, \( F_i \) is the area of cross section of the longitudinal bar, \( G_i \) is the elasticity modulus of the longitudinal bar in shear, \( I_{xy}, I_{xy} \) are the inertia moments of the cross-section of the longitudinal bar, \( k_i \) is the amount of longitudinal bars, \( q_i, q_j, q_k \) are the components of pressure force influencing on the cylindrical shell as viewed from medium, and
\[
\rho_i = \int \rho(z) \, dz, \quad \frac{1}{E_i} = \int \frac{\rho(z)}{E_i} \, dz
\]
It is considered that the rigid contact conditions between the shell and rings are satisfied:
\[
u_j (y) = u(x, y, z) + h j \phi_j (x, y)
\]
\[
\theta_j (x, y) = \theta_j (x, y) + h, \phi_j (x, y)
\]
\[
w_j (x, y) = w(x, y) - \phi_j (x, y)
\]
(9)
The system of motion of the medium in cylindrical coordinates are written as follows [8]:
\[
\left( \lambda_2 + 2 \mu_2 \right) \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{2 \mu_2}{r} \frac{\partial \phi}{\partial r} + 2 \mu_2 \frac{\partial^2 \phi}{\partial \phi^2} - \rho_2 \frac{\partial^2}{\partial t^2} = 0
\]
\[
\left( \lambda_2 + 2 \mu_2 \right) \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{2 \mu_2}{r} \frac{\partial \phi}{\partial r} + 2 \mu_2 \frac{\partial^2 \phi}{\partial \phi^2} - \rho_2 \frac{\partial^2}{\partial t^2} = 0
\]
(10)
\[
\left( \lambda_2 + 2 \mu_2 \right) \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{2 \mu_2}{r} \frac{\partial \phi}{\partial r} + 2 \mu_2 \frac{\partial^2 \phi}{\partial \phi^2} - \rho_2 \frac{\partial^2}{\partial t^2} = 0
\]
where, \( \lambda_2, \mu_2 \) are the components of displacement vector of medium, \( \lambda_2, \mu_2 \) are Lamé coefficients of medium, \( \rho_2 \) is the density of medium, \( r, \theta, \phi \) are longitudinal, radial and peripheral coordinates
\[
a = \frac{\lambda_2 + 2 \mu_2}{\rho_2}, \quad a_b = \frac{\mu_2}{\rho_2}
\]
Volume extension \( \theta \) and \( \omega_2, \omega_3, \omega_4 \) component are calculated by the following expressions:
\[
\theta = \frac{\partial s_3}{\partial r} + \frac{s_3}{r} + \frac{\partial s_3}{\partial \theta} + \frac{\partial s_3}{\partial \phi} + 2 \omega_3 = \frac{1}{r} \frac{\partial (r s_3)}{\partial r}
\]
\[
2 \omega_3 = \frac{\partial s_3}{\partial r} + \frac{s_3}{r} + \frac{\partial s_3}{\partial \theta} + \frac{\partial s_3}{\partial \phi}
\]
The stresses generated in medium are expressed by displacements \( s_2, s_3, s_4 \) as follows:
\[
\sigma_{xx} = \mu_2 \left( \frac{\partial s_3}{\partial r} + \frac{s_3}{r} \right)
\]
\[
\sigma_{yy} = \mu_2 \left( \frac{\partial s_3}{\partial r} + \frac{s_3}{r} + \frac{\partial s_3}{\partial \theta} + \frac{\partial s_3}{\partial \phi} \right)
\]
\[
\sigma_{rr} = \lambda_2 \left( \frac{\partial s_3}{\partial r} + \frac{s_3}{r} + \frac{\partial (r s_3)}{\partial r} + \frac{\partial s_3}{\partial \theta} + \frac{\partial s_3}{\partial \phi} + 2 \mu_2 \frac{\partial s_3}{\partial r} \right)
\]
(11)
When studying the vibrations of medium and a viscoelastic cylindrical shell stiffened with bars we will consider two cases: a) the inertial influence of medium on vibration process is weak; b) in studying the vibration process inertial influence of medium may not be ignored.

in the case a)
\[
s_x = \left[ \left( -kr \right) \frac{\partial I_n (kr)}{\partial r} - 4 (1 - v_s) k I_n (kr) \right] A_s + k I_n (kr) B_s \times \cos n \varphi \cos kx \sin \omega t
\]
\[
s_\theta = \left[ - \frac{n}{r} I_n (kr) B_s - \frac{\partial I_n (kr)}{\partial r} C_s \right] \times \sin n \varphi \sin kx \sin \omega t
\]
\[
s_r = \left[ -k^2 r I_n (kr) A_s + \frac{\partial I_n (kr)}{\partial r} B_s + n \frac{I_n (kr)}{r} C_s \right] \times \cos n \varphi \sin kx \sin \omega t
\]

in the case b)
\[
s_x = \left[ A_s k I_n (\gamma r) - \frac{C_s r^2}{\mu_s} I_n (\gamma r) \right] \times \cos n \varphi \cos kx \sin \omega t
\]
\[
s_\theta = \left[ - \frac{A_s n}{r} I_n (\gamma r) - \frac{C_s n k}{\mu_s} I_n (\gamma r) - \frac{B_s n}{\mu_s} \frac{\partial I_n (\gamma r)}{\partial r} \right] \times \sin n \varphi \sin kx \sin \omega t
\]
\[
s_r = \left[ A_s \frac{\partial I_n (\gamma r)}{\partial r} - \frac{C_s k}{\mu_s} \frac{\partial I_n (\gamma r)}{\partial r} + \frac{B_s n}{r} I_n (\gamma r) \right] \times \cos n \varphi \sin kx \sin \omega t
\]

The system of motion equations of medium (10) is complemented by contact conditions. We will assume that in the deformation process, the tangential surfaces of the cylindrical shell and medium shift with respect to another one but are not separated. In this case at the \( x = x_1 \) and \( x = x_2 \) the conditions \( \sigma_{xx} = 0; \sigma_y = 0 \) should be satisfied.

The condition of equality of normal displacement components
\[
s_x = u; s_\rho = \vartheta; s_\rho = w \quad (r = R) \quad (14)
\]
The conditions of equality of pressure forces
\[
q_x = -\sigma_{xx}; q_\rho = -\sigma_{y\rho}; q_\rho = -\sigma_{rr} \quad (r = R) \quad (15)
\]

Thus, the forces influencing on the cylinder as viewed from the medium may be determined by means of the obtained expressions. As a result, the solution of the stated problem is reduced to joint integration of total energy (8) of the construction consisting of medium-filled cylindrical shell stiffened with discretely distributed rings within the boundary conditions (14) and (15) of the system of motion Equations (10) of the medium.

III. PROBLEM SOLUTION

Represent the expressions of pressure components \( q_x, q_\rho, q_\rho \) in the following way:

\[
q_x = -\left( \hat{A}_s u_0 + \hat{B}_s \vartheta_0 + \hat{C}_s w_0 \right) \cos k \varphi \cos \frac{\pi x}{l} \sin \omega t
\]
\[
q_\rho = -\left( \hat{A}_s u_0 + \hat{B}_s \vartheta_0 + \hat{C}_s w_0 \right) \sin k \varphi \sin \frac{\pi x}{l} \sin \omega t
\]
\[
q_\rho = -\left( \hat{A}_s u_0 + \hat{B}_s \vartheta_0 + \hat{C}_s w_0 \right) \cos k \varphi \sin \frac{\pi x}{l} \sin \omega t
\]

In Equations (12), (13), (17), (18) \( A_s, B_s, C_s \) are unknown constant numbers, \( k, n, \gamma_s, \gamma_\rho \) are wave numbers, \( I_n \) is a modified, nth order, first kind Bessel function, \( \gamma_s^2 = k^2 - \mu^2 \), \( \gamma_\rho^2 = k^2 - \mu_s^2 \), \( k^* = kR \). \( \omega \) is unknown frequency. In Equation (16), as the coefficients of the unknowns \( \gamma^* = \gamma_R \), \( \gamma^*_s = \gamma_s R \), \( \mu^* = \mu_R \) are bulky, we do not cite them here \( u_0, \vartheta_0, w_0 \).

In Equation (8) \( u, \vartheta, w,T_1, T_2, M_1, M_2, S, H \) are variation quantities. Let’s determine stationary value of functional (8). Therefore, we use the Rietz method. We will look for unknown quantities in the following way:

\[
u = \cos \frac{\pi x}{l} \sin (k \varphi) (u_0 \cos \omega t + u_1 \sin \omega t)
\]
\[
\vartheta = \sin \frac{\pi x}{l} \cos (k \varphi) (\vartheta_0 \cos \omega t + \vartheta_1 \sin \omega t)
\]
\[
w = \sin \frac{\pi x}{l} \sin (k \varphi) (w_0 \cos \omega t + w_1 \sin \omega t)
\]
\[
T_1 = \sin \frac{\pi x}{l} \sin (k \varphi) (T_{10} \cos \omega t + T_{11} \sin \omega t)
\]
\[
T_2 = \cos \frac{\pi x}{l} \sin (k \varphi) (T_{20} \cos \omega t + T_{21} \sin \omega t)
\]
\[
S = \sin \frac{\pi x}{l} \sin (k \varphi) (S_{10} \cos \omega t + S_{11} \sin \omega t)
\]
\[
M_1 = \cos \frac{\pi x}{l} \sin (k \varphi) (M_{10} \cos \omega t + M_{11} \sin \omega t)
\]
\[
M_2 = \sin \frac{\pi x}{l} \sin (k \varphi) (M_{20} \cos \omega t + M_{21} \sin \omega t)
\]
\[
H = \cos \frac{\pi x}{l} \cos (k \varphi) (H_{10} \cos \omega t + H_{11} \sin \omega t)
\]

If we substitute the Equation (13) in functional (8), we get a function dependent on the variables \( u_0, u_1, \vartheta_0, \vartheta_1, w_0, w_1, T_{10}, T_{11}, T_{20}, T_{21}, S_{10}, S_{11}, M_{10}, M_{11}, M_{20}, M_{21}, S_{10}, S_{11}, H_{10}, H_{11} \). The stationarity condition of the obtained function is determined from following system:

\[
\sum_{i=1}^{10} \frac{\partial J}{\partial \varphi} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial u_i} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial \vartheta_i} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial w_i} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial T_{10}} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial T_{11}} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial T_{20}} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial T_{21}} = 0
\]
\[
\sum_{i=1}^{10} \frac{\partial J}{\partial S_{10}} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial S_{11}} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial M_{10}} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial M_{11}} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial H_{10}} = 0; \quad \sum_{i=1}^{10} \frac{\partial J}{\partial H_{11}} = 0
\]

(18)
As the system (18) is homogeneous, for the existence of its nonzero solution, the principal determinant should be equal to zero. As a result, we get the frequency equation:

\[
\det \begin{bmatrix} a_{ij} \end{bmatrix} = 0, \quad i, j = 1, 18
\]  

(19)

Equation (19) was studied by numerical method. The following values were taken for the parameters of medium and shell:

- \( h^* = \frac{h}{R} = 0.25 \times 10^{-2} \), \( \nu = 0.3 \); \( F_j = 5.75 \times 10^{-4} \) mm²;
- \( I_y = 19.9 \) mm⁴; \( I_{kj} = 0.48 \) mm⁴;
- \( E_j = 6.67 \times 10^9 \) H/m²;
- \( \alpha = 0.5 \); \( \rho_j = 0.26 \times 10^4 \) N.san²/m²;
- \( a_e = 2.25a_i \); \( a_i = 308 \) m/san; \( E_0 = E \); \( \rho_0 = \rho_j \).

Two cases of inhomogeneity functions were considered as linear

\[
E(z) = E_0 \left[ 1 + \alpha \left( \frac{z}{h} \right) \right], \quad \rho(z) = \rho_0 \left[ 1 + \alpha \left( \frac{z}{h} \right) \right]
\]

and parabolic

\[
E(z) = E_0 \left[ 1 + \alpha \left( \frac{z}{h} \right)^2 \right], \quad \rho(z) = \rho_0 \left[ 1 + \alpha \left( \frac{z}{h} \right)^2 \right].
\]

where, the Young modulus is \( \alpha \) nonhomogeneity parameter. Note that in the case of linear function \( |\alpha| < 1 \), in the case of parabolic principle \( \alpha \) is any number, and \( \omega_l = \sqrt{ \frac{(1-\nu^2)\rho_0 R^2 \alpha^2}{E} } \).

**IV. CONCLUSIONS**

The result of calculations was given in Figure 1 in the form of dependence of frequency parameter on compressive stress, in Figure 2 in the form of dependence of frequency parameter on the amount of rings. The line of nonhomogeneity laws corresponds to curves 1, parabolic change cases of nonhomogeneity laws correspond to curves 2. The calculations show that the number of vibration frequencies corresponding to linear case of inhomogeneity laws is greater than the number of vibration frequencies corresponding to the case of parabolic change. As is seen from Figure 1, as the value of compressive stress increases frequency of vibrations decreases and tends to zero. As is seen from Figure 2, the frequency of natural vibrations of the system at first increases and gets maximum value and again decreases. This is explained by the fact that as the number of rings increases, the mass also increases and inertial influence to the vibration process of the system amplifies.

**REFERENCES**

BIOGRAPHIES

Fuad Seyfeddin Latifov was born in Ismayilly, Azerbaijan, in 1955. He graduated from Faculty of Mechanics-Mathematics, Azerbaijan State University, Baku, Azerbaijan in 1977. In 1983, he received his M.Sc. degree in Physics-Mathematics from Saint Petersburg State University, Saint Petersburg, Russia. In 2003, he got the Ph.D. degree in Physics-Mathematics. He is a Professor and the Chief of Department of Higher Mathematics in Azerbaijan University of Architecture and Construction Baku, Azerbaijan. He has written more than 80 scientific articles and 11 monographs. He is the coauthor of an encyclopedia on mathematics.

Ramiz Aziz Iskanderov was born in Krasnoselo, Armenia on July 7, 1955. He received the M.Sc. degree in Mathematics-Mechanics from Azerbaijan (Baku) State University, Baku, Azerbaijan, in 1977. He also received the Ph.D. degree in Mathematics-Mechanics from Azerbaijan University of Architecture and Construction, Baku, Azerbaijan, in 1983. He is a Professor in the field of Mathematics-Mechanics in Department of Theoretical Mechanics, Azerbaijan University of Architecture and Construction since 1983. He has published 70 papers and one book. His scientific interests are theoretical mechanics, solid state mechanics and mechanics of composite materials.

Konul Agakhalaf Babayeva was born in Azerbaijan. She graduated from Faculty of Mechanics-Mathematics, Baku State University, Baku, Azerbaijan in 2008. She received her M.Sc. degree in Mathematical Analysis from Baku State University in 2011.