International Journal on "Technical and Physical Problems of Engineering" (IJTPE)
Published by International Organization of IOTPE
ISSN 2077-3528
IJTPE Journal
www.iotpe.com
ijtpe@iotpe.com
September 2017 | Issue 32 | Volume 9 | Number 3 | Pages 48-52

# FREE VIBRATIONS OF LATERAL REINFORCED CONICAL SHELL WITH SPRING ASSOCIATED MASS IN MEDIUM

**R.A. Iskanderov H. Shafiei Matanagh**

*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
*r.iskanderov@gmail.com, hossein_shafiei_m@yahoo.com*

**Abstract-** We consider a circular closed truncated medium-contacting conical shell of constant thickness regularly reinforced with lateral ribs supporting the associated mass connected with the shell in diametrically opposite points by means of two springs of the same rigidity.

**Keywords**: Conical Shell, Vibrations, Energetic Method, Winkler Model.

## I. INTRODUCTION

Conical shell is widely used in aviation engineering and machine building. One of the first papers on study of stability of conical shells was the paper of Kh.M. Mushtari [1]. In the references [2, 3] the equations of motion were obtained for conical shells reinforced with stiffening ribs under linear-elastic deformation with regard to transverse shifts. Mathematical model of deformation of general form reinforced orthotropic shells based on the functional of total energy of deformation was represented in the reference [4]. The reference [5] was devoted to construction of mathematical model of deformation of conical shell constructions based on the functional of total energy of deformation with regard to orthotropic of the material, geometrical nonlinearity and transversal shifts.

## II. PROBLEM STATEMENT

The problem of definition of natural frequencies of vibrations of such a shell is solved in linear statement by the energetic method. Discrete arrangement of reinforcing lateral ribs is taken into account.

Such a system of coordinates is accepted: variable radius $r$ and dihedral angle $\varphi$ between diametrical planes (Figure 1). The shell's potential energy is calculated by means of the following expression [10]:

$$\Pi_1=\frac{Eh}{12\left(1-\nu^2\right)}\int_{r_2}^{r_1}\int_0^{2\pi}\left(\varepsilon_1^2+\varepsilon_2^2+2\nu\varepsilon_1\varepsilon_2+\frac{1-\nu}{2}\psi^2\right)\frac{rdrd\varphi}{\sin\gamma}+$$
$$+\frac{D}{2}\int_{r_2}^{r_1}\int_0^{2\pi}\left(\chi_1^2+\chi_2^2+2\nu\chi_1\chi_2+2\left(1-\nu\right)\tau^2\right)\frac{rdrd\varphi}{\sin\gamma} \quad (1)$$

where according to [9] it is accepted that:

$$\varepsilon_1=\frac{\partial u}{\partial r}\sin\gamma\,,\ \varepsilon_2=\frac{u}{r}\sin\gamma+\frac{\partial \upsilon}{r\partial\varphi}+\frac{w}{r}\cos\gamma$$
$$\psi=\frac{\partial u}{r\partial\varphi}+\frac{\partial\upsilon}{\partial r}\sin\gamma-\frac{\upsilon}{r}\sin\gamma\,,\ \chi_1=-\frac{\partial^2 w}{\partial r^2}\sin^2\gamma$$
$$\chi_2=-\frac{u}{r^2}\sin\gamma\cos\gamma-\frac{w}{r^2}\cos^2\gamma-\frac{\partial^2 w}{r^2\partial\varphi^2}-\frac{\partial w}{r\partial r}\sin^2\gamma \quad (2)$$
$$\tau=-\frac{\cos\gamma}{r}\left(\frac{\partial u}{r\partial\varphi}-\frac{\partial\upsilon}{\partial r}\sin\gamma+\frac{\upsilon}{r}\sin\gamma\right)-\frac{2\sin\gamma}{r}\left(\frac{\partial^2 w}{\partial r\partial\varphi}-\frac{\partial w}{r\partial\varphi}\right)$$

where, $E$ is modulus of elasticity; $h$ is the shell's thickness; $\nu$ is the Poisson ratio; $r_1$, $r_2$ are the radii of the greater and less base of the shell; $\gamma$ is an angle between the generator and the axis of the shell; $u$, $\upsilon$, $w$ are the components of the vector of displacement of the points of the shell's medium surface along the generator in tangential direction and along the normal to the median surface;

$$D=\frac{Eh^3}{12\left(1-\nu^2\right)}.$$

Potential energy $\Pi_2$ of deformation of lateral ribs equals [6]:

$$\Pi_2=\frac{1}{2}\sum_{j=1}^{k_2}\int_0^{2\pi}\left[EF_2\left(\frac{\partial\vartheta}{r\partial\varphi}+\frac{w}{r}\right)+\right.$$
$$\left.+EI_2\left(\frac{\partial^2 w}{r^2\partial\varphi^2}+\frac{w}{r^2}\right)^2+GI_{2kp}\left(\frac{\partial^2 w}{r\partial r\partial\varphi}\sin\gamma\right)^2\right|_{r=r_j} \quad (3)$$

where, $F_2$, $I_2$, $I_{2kp}$ are area and moments of inertia of lateral cross section of the bar with respect to the axis that coincides with the generator, and also its inertia moment under torsion; $G$ is the shift modulus; $k_2$ is the amount of lateral bars; and $r_2$ is the coordinates of their location.

The kinetic energy $T_1$ of lateral ribs is calculated by means of the expression

$$T_1=\frac{\gamma_1 h}{2g}\int_{r_2}^{r_1}\int_0^{2\pi}\left[\left(\frac{\partial w}{\partial t}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial \vartheta}{\partial t}\right)^2\right]\frac{rdrd\varphi}{\sin\gamma}+$$
$$+\frac{\gamma_1 F_2}{2g}\sum_{j=1}^{k_2}\int_0^{2\pi}\left[\left(\frac{\partial w}{\partial t}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial \vartheta}{\partial t}\right)^2\right]_{r=r_j} rd\varphi \tag{4}$$

where, $\gamma_1$ is specific weight of the shell and ribs material; and $g$ is acceleration of gravitational force.

Influence of medium on a shell is determined as of external surface loads applied to the shell and is calculated as a work done by loads when taking the system from deformed state to initial undeformed one and is represented in the form:

$$A=\int_{r_2}^{r_1}\int_0^{2\pi} q_r rdrd\varphi \tag{5}$$

We will take into account the motion of mass only in the plane of cross section along the axis $z$ (Figure 1). The motion of the mass from this plane and deformation of springs caused by displacements of the points of their fastening to the shell in the direction of displacement vector $u$ is ignored. As the problem is solved in linear statement, i.e. in der assumption that the displacement of the points of the deformable system is small, such assumptions are rightful.

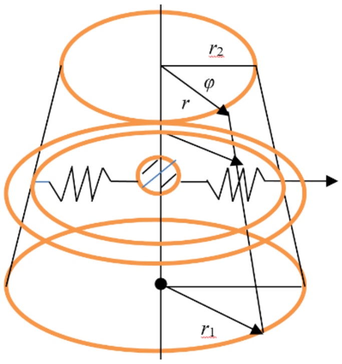


Figure 1. Lateral reinforced, medium-contacting conical shell with mass

Potential energy $\Pi_3$ of springs and kinetic energy $T_2$ of the mass are equal to $\Pi_3=\left(z-w_0 cos\gamma\right)^2 c$; $T_2=\frac{1}{2}M\dot{z}^2$, respectively, where $w_0$ is the displacement of the spring's fastening points, arranged in diametrical coordinate plane $\varphi=0$.

## III. PROBLEM SOLUTION

We represent displacements of the shell as follows [7]:

$$w=\frac{r^2}{r_1^2}\sin\frac{m\pi\left(r-r_2\right)}{r_1-r_2}\sum_{n=1}^{\infty}A_n\left(t\right)\cos n\varphi$$
$$\vartheta=\frac{r^2}{r_1^2}\sin\frac{m\pi\left(r-r_2\right)}{r_1-r_2}\sum_{n=1}^{\infty}B_n\left(t\right)\sin n\varphi \tag{6}$$
$$u=\frac{r^2}{r_1^2}\cos\frac{m\pi\left(r-r_2\right)}{r_1-r_2}\sum_{n=1}^{\infty}D_n\left(t\right)\cos n\varphi$$

where, $n$ is the amount of waves in peripheral direction; and $m$ is the amount of half waves along the generator.

These expressions satisfy such conditions of elastic built-in of the shell along the edges under which $w=\vartheta=0$. Equation (5) are convenient in the sense that owing to the multiplier $r^2$, the integrals encountered when calculating the energy, are taken in quadrature and this simplifies the problem solution. Besides this, Equation (6) qualitatively reflect the fact that the crests of half-waves of the form of vibrations along the generator of the conical shell are somewhat displaced to the side of its greater base.

It should be noted that for even $n$ the springs are deformed, the mass does not move; for odd $n$ the mass performs vibrations along the axis $z$. In the sequel we will consider this case.

Substituting the found expressions for kinetic and potential energy in Lagrange's known equation of second kind [8] we get the system of ordinary differential equations infinite with respect to $n$:

$$M\ddot{z}+2c(z-\alpha_0\sum_{n=1,3,\ldots}^{\infty}A_n)=0$$
$$2\ddot{A}_1 a+2A_1\left(l_1+\alpha_0^2 c+\beta_0\right)+B_1 f_1+D_1 q_1-$$
$$-2z\alpha_0 c+2c\alpha_0^2\left(A_3+A_5+A_7+\ldots\right)=0$$
$$2\ddot{B}_1 a+2B_1 d_1+A_1 f_1+D_1 s_1=0$$
$$2\ddot{D}_1 b+2D_1 e_1+A_1 q_1+B_1 s_1=0 \tag{3}$$
$$2\ddot{A}_3 a+2A_3\left(l_3+\alpha_0^2 c+\beta_0\right)+B_3 f_3+D_3 q_3-$$
$$-2z\alpha_0 c++2c\alpha_0^2\left(A_1+A_3+A_7+\ldots\right)=0$$
$$2\ddot{B}_3 a+2B_3 d_3+A_3 f_3+D_3 s_3=0$$
$$2\ddot{D}_3 b+2D_3 e_3+A_3 q_3+B_3 s_3=0$$

where,

$$l_n=\Omega_1 R_1\cos^2\gamma+\frac{h^3}{3}\left\{\sin^4\gamma\left(\begin{array}{l}\frac{117}{8}R_2+\\ \frac{31}{32}\alpha^2 R_4+\\ \frac{1}{192}\alpha^4 R_6\end{array}\right)+\right.$$
$$+\frac{1}{4}R_2\left[\left(1-n^2\right)^2-\sin^4\gamma\right]+\frac{1}{8}\left(\frac{1}{4}\alpha^2 R_4+3R_2\right)\sin^4\gamma+$$
$$+\nu\left[\frac{1}{2}\alpha^2 R_4\sin^4\gamma-R_2\sin^4\gamma-R_2\left(1-n^2+\sin^2\gamma\right)\sin^2\gamma-\right.$$
$$\left.-2\left(1-n^2+\sin^2\gamma\right)\frac{1}{32}\alpha R_4-\frac{3}{2\alpha}R_2\right]-$$
$$\left.-(1-\nu)\left(7R_2-\frac{1}{4}R_4\alpha\right)n^2\sin^2\gamma\right\}+$$
$$+\frac{\pi}{2r_1^4}\sum_{j=1}^{k_2}\left[EF_2 r_j^4+EI_2\left(n^2-1\right)+\right.$$
$$\left.+GI_{kp}n^2\left(2\sin\theta_j-\frac{1}{2}\alpha r_j\cos\theta_j\right)^2\right]r_j\sin\theta_j;$$

$$e_n = \frac{1}{2}\Omega_1^2 \sin^2\gamma \left\{ \frac{1}{\alpha^2} R_2 \left[ 9 - 12\nu + \frac{3(1-\nu)n^2}{4} \right] + \right.$$
$$+ R_4 \left[ \nu + \frac{7}{8} + \frac{1}{16}(1-\nu)n^2 \right] + \frac{1}{48}\alpha^2 R_6 \Big\} +$$
$$+ \frac{h^3}{96}\Omega_1 \left[ \frac{1}{4}\sin^2 2\gamma + 2(1-\nu)n^2\cos^2\gamma \right] R_2 ;$$
$$d_m = \Omega_1 \left\{ \frac{1}{2}n^2 \left( \frac{1}{8}R_4 - \frac{3}{2\alpha^2}R_2 \right) + \right.$$
$$+ \frac{1-\nu}{2}\left( \frac{1}{4}R_4 - \frac{3}{\alpha^2}R_2 + \frac{\alpha^2}{48}R_6 \right) \Big] +$$
$$+ \frac{h^2(1-\nu)\sin 2\gamma}{12}\frac{1}{128}R_4 - \frac{11}{32}R_2 \Big\} + \frac{EF_2 n^2}{2r_1^4}\sum_{j=1}^{k_2}\sin^2\theta_j ;$$
$$f_n = -\Omega_1 \left[ \left( \frac{1}{8}R_4 - \frac{3}{2\alpha^2}R_2 \right) n\cos\gamma + \right.$$
$$+ \frac{1}{12}h^2 n \sin 2\gamma (1-\nu)\left( \frac{9}{4}R_2 + \frac{1}{16}\alpha^2 R_4 \right) \Big] +$$
$$+ \frac{\pi n E F_2}{r_1^4}\sum_{j=1}^{k_2} r_j^3 \sin^3\theta_j ;$$
$$q_n = \Omega_1 \frac{1}{4}\sin 2\gamma \left( \Omega + \frac{\nu\alpha R_5}{10} \right) -$$
$$- \frac{h^2}{24}\left\{ \left(1 - n^2 + \sin\gamma\right)\frac{1}{\alpha}R_2 + \left[ \frac{\alpha}{4}R_3\left( \nu + \frac{1}{4} \right) + \right. \right. ;$$
$$+ \frac{1}{\alpha}R_4(1+5\nu) + \Omega\nu\alpha^2 \Big] \frac{1}{2}\sin^2\gamma +$$
$$+ \frac{\alpha}{8}R_3(1-\nu)n^2 \Big\} \sin 2\gamma ;$$
$$s_n = -\frac{1}{4}\Omega_1 \left\{ \begin{array}{l} \frac{h^2(1-\nu}{192} n\alpha R_3 \sin^2 2\gamma + \\ \Omega\left[ 3n + (4-n)\nu \right] + \frac{\pi\alpha}{10}R_5(1+\nu) \end{array} \right\} ;$$
$$a = \beta_1 - \beta_2; \quad b = \beta_1 + \beta_2;$$
$$\beta_1 = \frac{\gamma_1}{2gr_1^4}\left( \frac{\pi h R_6}{12\sin\gamma} + \pi F_2 \sum_{j=1}^{k_2} r_j^5 \sin^2\theta_j ; \right) ;$$
$$\beta_2 = \frac{\gamma_1}{2gr_1^4}\frac{\pi h}{\sin\gamma}\left( \frac{5}{2\alpha^2}R_4 + \frac{30}{\alpha^4}R_2 \right) ;$$
$$\Omega = \frac{r_1^3}{\alpha} - \frac{6r_1}{\alpha^3} - \left( \frac{r_2^3}{\alpha} - \frac{6r_2}{\alpha^3} \right) ;$$
$$\Omega_1 = \frac{Eh\pi}{\left(1-\nu^2\right) r_1^4 \sin\gamma} \; ; \; \alpha = \frac{2\pi m}{r_1 - r_2} \; ;$$
$$R_p = r_1^p - r_2^p \; (p = 1, \ldots 6) ;$$
$$\beta_0 = \frac{k\pi}{r_1^2}\left\{ \frac{1}{12}\left( r_2^6 - r_1^6 \right) + \frac{5(r_2 - r_1)\left(r_1^2 - r_2^2\right)}{4m\pi} r_1^2 + r_2^2 - 12 \right\} ;$$

Accepting the solution of the infinite system in the form $z = z_n^* \sin\omega t$, $A_n = A_n^* \sin\omega t$, $B_n = B_n^* \sin\omega t$, $D_n = D_n^* \sin\omega t$, where $\omega$ is the natural frequency of vibrations of the studied system, and substituting these solutions in Equation (3), we get an infinite homogeneous system of algebraic equations with respect to unknowns $z_n^*$, $A_n^*$, $B_n^*$, $D_n^*$.

Assuming that the influence of the medium on the shell is subjected to the Winkler model, i.e. $q_r = kw$, we can represent (2) in the form:

$$A = \int_{r_2}^{r_1}\int_0^{2\pi} k\left(1 - \beta\omega^2\right) w^2 \; r dr d\varphi \quad (4)$$

where, $k$, $\beta$ are constants.

Equate to zero the determinant of this system

$$\begin{vmatrix} \lambda_1 + \lambda^* & R & R & \ldots \\ R & \lambda_3 + \lambda^* & R & \ldots \\ R & R & \lambda_5 + \lambda^* & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0 \quad (5)$$

where,

$$\lambda_n = \frac{l_n}{a} + \frac{f_n}{2a\Delta}\left[ \frac{q_n s_n}{4ab} - \frac{f_n}{2a}\left( \frac{e_n}{b} - \omega^2 \right) \right] +$$
$$+ \frac{q_n}{2a\Delta}\left[ \frac{f_n s_n}{4ab} - \frac{q_n}{2ab}\left( \frac{d_n}{a} - \omega^2 \right) \right] ;$$
$$\lambda^* = R - \omega^2 + \beta_0 / 2a ;$$
$$\Delta = \left( \frac{d_n}{a} - \omega^2 \right)\left( \frac{e_n}{b} - \omega^2 \right) - \frac{s_n^2}{4ab} ;$$
$$R = -\frac{\alpha_0^2\omega^2}{a}\left( \frac{c}{\frac{c}{M} - \omega^2} \right) .$$

As a result, we get an equation that allows to determine natural frequencies of vibrations of the medium-filled shell with associated mass. The influence of the associated mass is included into the expression for $R$. Consider the limit cases. For $c = 0$ and $R = 0$ we get a frequency equation for a medium-filled shell without associated mass $\lambda_n + R - \omega^2 + \frac{\beta_0}{2a} = 0$.

For $M = 0$ the results are similar. If $c = \infty$, then $R = -\frac{\alpha_0^2 M\omega^2}{a}$ mass is associated to the shell by means of absolutely rigid bars; for $M = \infty$, $R = \frac{\alpha_0^2 c}{a}$ is a shell with inner elastic bonds.

Natural frequencies of the system's vibrations were determined by numerical solution of Equation (5). The followings for accepted for the problem parameters: $r_1 = 160$ mm, $r_2 = 85$ mm, lateral bar of angle profile with dimensions $6 \times 10 \times 1$ mm, $k_2 = 2$, $m = 1$,

$h = 0.5$ mm. The lateral bars were fastened to the external surface of the shell. The associated mass whose quantity varies in the research process, was fastened in the middle of the shell in diametrically opposite points.

## IV. CONCLUSIONS

The curve refracting the dependence of natural frequency $f^* = \frac{\omega}{2\pi}$ of vibrations of medium-filled shell with no associated mass on the number of waves $n$ in peripheral direction is depicted in Figure 2. It is seen that with increasing the number of waves $n$ in peripheral direction, the frequencies of vibrations of medium-filled shell with no associated mass at first decreases, attains minimum and begins to increase.

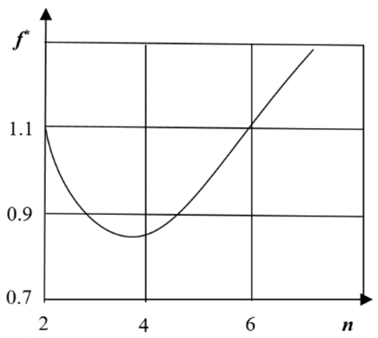


Figure 2. Dependence of natural frequency of shell's vibrations on the number of waves in peripheral direction

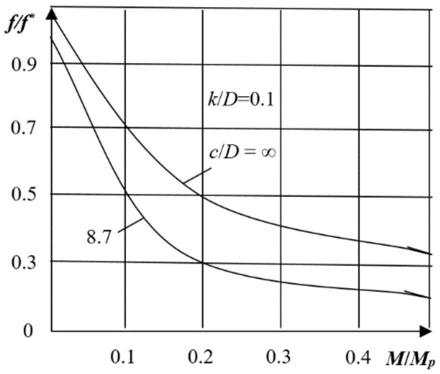


Figure 3. Dependence of natural frequency of shell's vibrations on the quantity of associated mass

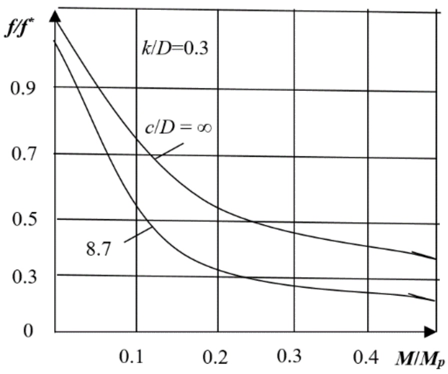


Figure 4. Dependence of natural frequency of shell's oscillations on the quantity of associated mass

The curves reflecting the dependence of minimum natural frequency of the system's vibrations on quantity of associated mass, relative rigidity of springs $\bar{c} = \frac{c}{D}$ for relative bed coefficient $\bar{k} = \frac{k}{D} = 0.1$, were depicted in Figure 3. The similar dependence for $\bar{k} = \frac{k}{D} = 0.3$ is in Figure 4. Analysis of curves show that influence of associated mass on minimum natural frequency of shell's vibrations is very essential. With decreasing the rigidity of bonds between the mass and shell, this influence increases. Comparison of curves from Figures 3 and 4 show that with increasing relative bed coefficient, minimum natural frequency of the systems vibration increases. This is connected with the fact that the Winkler model does not take into account inertial actions of medium on vibrations process.

## REFERENCES

[1] Kh.M. Mushtaru, "On Stability of Thin Shelled Conical Shells Circular of Cross-Section in Torsion by Pairs", Collection of Scientific Works of CAI, Kazan Aviation Institute, Russia, pp. 39-40, 1935.

[2] A.A. Ovcharov, "Mathematical Model of Conical Shell of Graduated-Variable Thickness under Dynamical Loading", Mathematic Modeling, Numerical Methods and Program Complexes, pp. 127-132, 2004.

[3] A.A. Ovcharov, "Computer Technologies of Studying Stability of Panels of Ridge Conical Shells", Herald of Civil Engineers, Vol. 11, No. 2, pp. 104-111, 2007.

[4] V.V. Karpov, "Mathematical Model of Deformation of Reinforced Orthotropic Rotation Shells", Engineering and Construction Magazine, No. 5, pp. 100-106, 2013.

[5] A.A. Semenov, A.A. Ovcharov, "Mathematical Model of Deformation of Orthotropic Conical Shells", Engineering Journal, Dona, Vol. 29, No. 2, pp. 45-50, 2014.

[6] I.Ya. Amiro, V.A. Zarutsky, P.S. Polyakov, "Ridge Cylindrical Shells", Naukova Dumka, Kiev, Ukraine, p. 245, 1973.

[7] L.M. Bunich, O.M. Paliy, I.A. Piskovitina, "Stability of Truncated Conical Shell under the Action of Uniform External Pressure", Engineering Collection, Vol. 23, pp. 89-93, 1956.

[8] N.N. Bucholtz, "Basic Course of Theoretical Mechanics", Science, Moscow, Russia, p. 467, 1972.

[9] V.Z. Vlasov, Selected Papers in 2 Volumes, Publication House, Vol. 1, Moscow, Russia, p. 528, 1962.

[10] S.P. Timoshenko, "Stability of Elastic Systems", QITTL, Moscow, Russia, p. 567, 1955.

[11] V.G. Palamarchuk, A.M. Nosachenko, "Free Vibrations of Ridge Conical Shell with Spring Associated Mass", Applied Mechanics, Vol. XVI, No. 4, pp. 40-46, 1978.

## BIOGRAPHIES

**Ramiz Aziz Iskanderov** was born in Krasnoselo, Armenia on July 7, 1955. He received the M.Sc. degree in Mathematics-Mechanics from Azerbaijan (Baku) State University, Baku, Azerbaijan, in 1977. He also received the Ph.D. degree in Mathematics-Mechanics from Azerbaijan University of Architecture and Construction, Baku, Azerbaijan, in 1983. He is a Professor in the field of Mathematics-Mechanics in Department of Theoretical Mechanics, Azerbaijan University of Architecture and Construction. He has published more than 70 papers and one book. His scientific interests are theoretical mechanics, solid state mechanics and mechanics of composite materials.

**Hossein Shafiei Matanagh** was born in Bostanabad, Iran, 1970. He received the B.Sc. and M.Sc. degrees from Azerbaijan Architecture and Construction University, Baku, Azerbaijan in 2012 and 2015, respectively. At present he is a Ph.D. student at the same university. He has published 3 scientific papers.