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OSCILLATION CHARACTERISTICS OF NON-HOMOGENEOUS BEAMS WITH VARIABLE CROSS-SECTION ON VISCOELASTIC FOUNDATIONS

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Abstract- In the presented article, the problem of free oscillatory motion of a beam with different cross-sectional dimensions located on a non-homogeneous, two-constant (Pasternak-type) viscoelastic base is investigated. During the study, the different mechanical properties of the beam material in tension and compression, the variation of the cross-section along the length, and the non-uniformity of the mechanical properties of the base were modeled in a complex way. In this work, first, an analytical expression of the bending moment for a beam of variable cross-section was obtained using the condition of the absence of longitudinal force. Then, the equation of motion was formulated taking into account the resistance of the non-uniform viscoelastic base. Since the obtained equation of motion is a complex differential equation with a particular derivative due to bending, its solution was carried out by approximate analytical methods. To solve a fourth-order differential equation with variable coefficients, the separation of variables method was applied and an approximate analytical solution was obtained using the Bubnov-Galerkin method. As a result of the study, functional dependence equations were obtained between the parameters characterizing the inhomogeneity of the base and material at a dimensionless circular frequency, as well as the change in the cross section. The numerical calculations performed showed that the increase in the inhomogeneity parameters and the resistance of the external environment has a significant effect on the frequency value. The results are presented in the form of graphs and corresponding dependence equations. The results obtained allow for a more accurate assessment of the dynamic behavior of beams with different cross-sectional dimensions and different moduli and are of practical importance in calculating their strength, stability, and frequency characteristics.

Keywords: Compression, Bending, Elasticity, Stability, Free Oscillation, Beam, Inhomogeneous, Dimensionless Frequency.

1. INTRODUCTION

The dynamic behavior of systems of beams with variable cross-sections and different moduli is of great interest in engineering and applied mechanics. Such systems are found in many engineering applications: bridge and pier structures, marine platforms and offshore structures, rotating shafts of drilling rigs and other mechanical components. In these structures, both the geometric parameters of the beam and the mechanical properties of the base can vary along the length, i.e. the system is non-uniform and inhomogeneous.

The dynamic analysis of structures with such non-uniformity and variable parameters is of great importance from an engineering point of view, since classical beam models with constant parameters do not fully reflect the behavior of real systems. Traditional beam models, such as the Euler–Bernoulli and Timoshenko beam theories, are often based on the assumptions of homogeneous material and constant cross-section. At the same time, the effect of the elastic medium is usually taken into account using a simplified Winkler-type model. However, these approaches do not include the internal interaction and inhomogeneity of the foundation, and as a result, the natural frequencies and deformation modes of the system do not fully reflect the real behavior for the engineer. In such cases, the Pasternak-type viscoelastic foundation model is considered more appropriate, since it takes into account both the elastic response and the interaction between the points of the foundation.

The cross-sectional dimensions of variable-section beams vary along their length. Therefore, their mechanical properties change. At the same time, the change in cross-sectional dimensions affects the vibration properties of the beam.

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In [1], [2] and [3] have analyzed in detail the stress-strain dependences for materials exhibiting different mechanical behavior in tension and compression. In addition, the oscillatory motion characteristics of a shaft placed on an elastic base characterized by two constants have also been investigated in those works. In [4] and [5], the stability problems of beams with different elastic moduli in tension and compression, as well as rectangular plates, were extensively studied.

In [6], a new theoretical approach was proposed for calculating the factors characterizing the resistance of the external environment. Unlike the classical model proposed by Winkler, here not only the vertical stiffness but also the resistance factor to sliding was taken into account. [7] and [8] have investigated the oscillatory motion problems of structural elements placed on a viscoelastic base. However, in those works, the elastic moduli were assumed to be the same. In [9-12], he considered the issues of vibrations in a viscoelastic medium when the heterogeneity of the cylindrical shell material changes in three directions. The results have been carefully analyzed.

Since materials with different resistances in tension and compression are widely used in practice, the solution of similar problems taking this property into account is of particular relevance. The fact that the beam material has different elastic moduli, the cross-section changes along the length, and at the same time the resistance of the external environment is taken into account complicates the mathematical formulation of the problem and complicates the analysis of the obtained results.

Thus, the presented model allows for a more accurate description of the dynamic behavior of real engineering systems, unlike classical beam models, and also provides a practical tool for evaluating the influence of various structural and material parameters. This approach is of great importance for engineers, especially in the design and analysis of variable cross-section beam systems located on non-homogeneous and non-homogeneous elastic bases.

2. PROBLEM STATEMENT

The longitudinal strain (ε) at any point in a beam during bending is equal to the sum of the longitudinal strain (l_0) of the neutral axis and the strain due to bending of the beam. According to the hypothesis of straight sections (Bernoulli's hypothesis), the section remains flat even after bending. In this case, the total longitudinal strain at any point in the section is expressed as:

$$\varepsilon = l_0 - \eta z_t \tag{1}$$

where, l_0 is Longitudinal deformation of the center line (axis) of the beam, η is Curvature of the beam, z_t is the distance of the observed point from the neutral axis. According to the classical Hooke's law, there is a linear relationship between stress and deformation ($\sigma = \varepsilon E$). However, since the material exhibits different properties in tension and compression, the modulus of elasticity (E) is different in the tension and compression zones of the cross section.

The stress in the part of the beam subjected to tension above (or below - depending on the sign convention) the neutral axis:

$$\sigma_h^+ = E_h^+ \varepsilon \tag{2}$$

If we substitute the deformation expression above, we get the following equation:

$$\sigma_h^+ = E_h^+ \left(l_0 - \eta z_t\right) \ , \ z_t \in A_1 \tag{3}$$

The resistance of the material in the section subjected to compression is characterized by a different modulus (E_h^-):

$$\sigma_h^- = E_h^- \varepsilon$$

If we add the deformation expression, we get Equation (5):

$$\sigma_h^- = E_h^- \left(l_0 - \eta z_t\right) \ \ z_t \in A_1 \tag{4}$$

These equations combine two important factors from an engineering perspective:

- Geometric factor: The $(l_0 - \eta z_t)$ part shows how the cross section deforms.
- Physical factor: The coefficients E_h^+ and E_h^- - represent the non-uniformity of the internal resistance of the material.

In materials with different modulus (E_h^+, E_h^-), this distribution of stresses causes the neutral axis to deviate from the geometric center of the cross section. Using the condition that the longitudinal force is equal to zero, the dependence between l_0 and η is found with the help of these two equations. Here E_h^+, E_h^- – are the modulus of elasticity in tension and compression for the shaft material, A_1 and A_2 are the tension and compression zones.

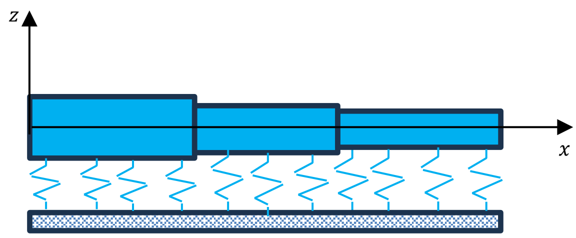

Figure 1. Beam resting on a non-uniform viscous elastic base

Let us write the equations of equilibrium and neutral axis for a shaft of variable cross-section. The boundary of the neutral axis z_0 is calculated as follows:

$$l_0 - \eta z_t = 0 \tag{5}$$

Using the condition of no force acting on the shaft length, we can write:

$$\int_{-t(x)}^{z_t} \sigma_h^+ \times dz_t + \int_{z_t}^{t(x)} \sigma_h^- \times dz_t = 0 \tag{6}$$

According to expression (6), we write the bending moment equation:

$$T = a\left[\int_{-t(x)}^{z_t} \sigma_h^+ \times z_t dz_t + \int_{z_t}^{t(x)} \sigma_h^- \times z_t dz_t\right] \tag{7}$$

3. PROBLEM SOLUTION

In the article, the base on which the beam is placed is assumed to be a Pasternak-type inhomogeneous viscoelastic medium. In this case, the reaction force exerted by the base on the beam is written as follows:

$$F = m_1(x).f - m_2(x).\partial^2 f / \partial t^2 \tag{8}$$

where, $m_1(x)$, $m_2(x)$ are continuous functions that take into account the characteristics of the viscous elastic base, determined experimentally. Non-homogeneity is described by the following analytical functions:

$$\bar{m}_1(x) = (1+\rho\bar{x})L_1 \tag{9}$$

$$\bar{m}_2(x) = (1+\rho\bar{x})L_2 \tag{10}$$

where, R_1, R_2 are the initial parameters of the basis, β is the non-homogeneity parameter. Two of the most commonly used models in modeling structures on elastic foundations are the Winkler and Pasternak-type foundation models. The main advantages of the Winkler model are its mathematical simplicity and its use for initial evaluation in many engineering calculations.

However, the main disadvantage of this model is that the internal interaction of the foundation is not taken into account. That is, the deformation occurring at one point of the foundation does not affect other points, which does not fully reflect real engineering systems. The Pasternak model is an advanced form of the Winkler model and allows describing a more realistic mechanical behavior of the elastic medium. In this model, the response of the foundation is determined not only by local deformation, but also by the interaction between neighboring points of the foundation. For this purpose, the Pasternak model was applied in the article in the application of the problems of dancing motion. Taking into account the resistance of the foundation, using the dynamic equilibrium condition, we obtain the equilibrium equation for a beam with different cross-sectional dimensions:

$$\begin{aligned} &E_t^+ J_0 \frac{\partial^2}{\partial x^2}\left[A(x)\frac{\partial^2 f}{\partial x^2}\right] + R_1 f - \\ &-R_2\frac{\partial^2 f}{\partial x^2} + \eta_0 t(x)\frac{\partial^2 f}{\partial t^2} = 0 \end{aligned} \tag{11}$$

Let us first examine Equation (6). Taking (5) into account, Equations (6) and (7) can be written as follows:

$$\int_{-t(x)}^{z_t}(l_0 - \eta z_t)dz_t + \mu\int_{z_0}^{h(x)}(l_0 - z_t\eta)dz_t = 0 \tag{12}$$

$$\left[\int_{-t(x)}^{z_t}(l_0 - z_t\eta)z_t dz_t + E^-\int_{z_t}^{t(x)}(l_0 - z_t\eta)z_t dz_t\right] = \frac{T}{E^+ a} \tag{13}$$

If we take the constants outside the integral, equation (12) becomes:

$$\begin{aligned} &l_0\left(\int_{-t(x)}^{z_t} dz_t + z_t \times \int_{z_t}^{+t(x)} dz_t\right) - \\ &-\eta\left(\int_{-t(x)}^{z_t} z_t dz_t + \mu \times \int_{z_t}^{+t(x)} z_t dz_t\right) = 0 \end{aligned} \tag{14}$$

From here we calculate the deformation of the centerline of the shaft:

$$l_0 = \eta \frac{\int_{-t(x)}^{z_t} z_t dz_t + \mu\int_{z_0}^{t(x)} z_t dz_t}{\int_{-t(x)}^{z_t} dz_t + \mu\int_{z_t}^{+t(x)} dz_t} \tag{15}$$

From Equation (15) we obtain the following expression:

$$\begin{aligned} &\frac{T}{E^+ a} = l_0\left(\int_{-t(x)}^{z_t} z_t dz_t + \int_{z_t}^{+t(x)} z_t dz_t\right) - \\ &-\eta\left(\int_{-t(x)}^{z_t} z_t^2 dz_t + \mu\int_{z_t}^{+t(x)} z_t^2 dz_t\right) \end{aligned} \tag{16}$$

Considering expression (15) in Equation (16), we can write:

$$\begin{aligned} &\frac{T}{E^+ a} = \eta \times \left[\frac{\left(\int_{-t(x)}^{z_t} z_t dz_t + \mu\int_{z_t}^{+t(x)} z_t dz_t\right)^2}{\int_{-t(x)}^{z_t} dz_t + \mu\int_{zt}^{+t(x)} dz_t} - \right. \\ &\left. -\left(\int_{t(x)}^{z_t} z_t^2 dz_t + \varpi\int_{z_t}^{+t(x)} z_t^2 dz_t\right)\right] \end{aligned} \tag{17}$$

Let us adopt the following notation in equation (17):

$$L = \frac{1}{I_0}\left[\frac{\left(\int_{-t(x)}^{z_T} z_t dz_t + \mu\int_{z_t}^{+t(x)} z_t dz_t\right)^2}{\int_{-t(x)}^{z_t} dz_t + \mu\int_{z_t}^{+t(x)} dz_t} - \left(\int_{t(x)}^{z_t} z^2 dz + \alpha\int_{z_t}^{+t(x)} z_t^2 dz_t\right)\right] \tag{18}$$

The bending stiffness $E^+ I_0$ $(I = at^3/12)$ of a shaft. Considering (18) in (17), we can write:

$t(x) = a_0^x(1+\gamma_1 x/l)$, $\varepsilon \in [0,1]$ are assumed. Let's calculate the expression $K(x)$.

Let us calculate the $\int_{-a_0^x(1+\gamma_1\bar{x})}^{+a_0^x(1+\gamma_1\bar{x})} dz_t$, $\int_{-a_0^x(1+\gamma_1\bar{x})}^{+a_0^x(1+\gamma_1\bar{x})} z_t dz_t$, $\int_{-a_0^x(1+\gamma_1\bar{x})}^{+a_0^x(1+\gamma_1\bar{x})} z_t^2 dz_t$, integrals given in expression (17):

$$\begin{aligned} &\int_{-a_0^x(1+\gamma_1\bar{x})}^{+a_0^x(1+\gamma_1\bar{x})} dz_t = \int_{-a_0^x(1+\gamma_1\bar{x})}^{z_t} dz_t + \mu\int_{z_0}^{+b_0^x(1+\varepsilon_1\bar{x})} dz_t = z_t + \\ &+a_0^x(1+\gamma_1\bar{x}) + \mu\left[a_0^x(1+\mu_1\bar{x}) - z_0\right] = \\ &= z_t(1-\mu) + a_0^x(1+\mu_1\bar{x}) \\ &\int_{-a_0^x(1+\gamma_1\bar{x})}^{+a_0^x(1+\gamma_1\bar{x})} dz_t = z_t(1-\mu) + a_0^x(1+\gamma_1\bar{x}) \end{aligned} \tag{19}$$

$$\int_{-a_0^x(1+\gamma_1\bar{x})}^{z_t} z_t dz_t + \mu \int_{z_t}^{+a_0^x(1+\gamma_1\bar{x})} z_t dz_t = \frac{1}{2}\left[z_t^2 - a_0^2\left(1+\gamma\bar{x}\right)\right] + \frac{\mu}{2}\left[a_0^2\left(1+\gamma_1\bar{x}\right)^2 - z_t^2\right] = \frac{1}{2}z_t^2\left(1-\mu\right) - \frac{a_0^2}{2}\left(1-\mu\right)\left(1+\gamma_1\bar{x}\right)^2 \quad (20)$$

$$\int_{-a_0(1+\gamma_1\bar{x})}^{z_t} z_t^2 dz_t + \mu \int_{z_t}^{+a_0(1+\gamma\bar{x})} z_t^2 dz_t = \frac{1}{3}\left[z_t^3 - a_0^3\left(1+\gamma_1\bar{x}\right)^3\right] + \frac{\mu}{3}\left[a_0^3\left(1+\gamma_1\bar{x}\right)^3 - z_t^3\right] = \frac{1}{3}z_t^3\left(1-\mu\right) - \frac{1}{3}\left(1+\mu\right)a_0^3\left(1+\gamma_1\bar{x}\right)^3 \quad (21)$$

For a beam with the same elastic modulus, the integrals take the following values:

$$\int_{-a_0(1+\gamma_1\bar{x})}^{+a_0(1+\gamma_1\bar{x})} dz_t = 2a_0\left(1+\gamma_1\bar{x}\right)$$
$$\int_{-a_0(1+\gamma_1\bar{x})}^{a_0(1+\gamma_1\bar{x})} z_t dz_t = 0 \quad (22)$$
$$\int_{-a_0(1+\gamma_1\bar{x})}^{+a_0(1+\gamma_1\bar{x})} z_t^2 dz_t = -\frac{2}{3}a_0^3\left(1+\gamma_1\bar{x}\right)^3$$

For a single-module constant-height beam, the integrals (20) and (21) take the following values:

$$\int_{-a_0}^{+a_0} dz_t = 2a_0$$
$$\int_{-a_0}^{a_0} z_t dz_t = 0 \quad (23)$$
$$\int_{-a_0}^{+a_0} z_t^2 dz_t = -\frac{2}{3}a_0^3$$

To simplify the calculations, we will neglect the terms involving γ^2 and γ^3, since they are much smaller compared to the other terms. Equation (19) remains unchanged. The terms in Equations (20) and (21) can be written as follows:

$$\int_{-a_0(1+\gamma_1\bar{x})}^{+a_0(1+\gamma_1\bar{x})} z_t dz_t \approx \frac{1}{2}z_t^2\left(1-\mu\right) - \frac{a_0^2}{2}\left(1-\mu\right)\left(1+2\gamma_1\bar{x}\right)$$
$$\int_{-a_0(1+\gamma_1\bar{x})}^{+a_0(1+\gamma_1\bar{x})} z_t^2 dz_t \approx \frac{1}{3}z_t^3\left(1-\mu\right) - \frac{a_0^3}{3}\left(1-\mu\right)\left(1+3\gamma_1\bar{x}\right) \quad (24)$$

The expressions for the $A(x)$ function are written as follows:

For a single-modulus beam:

$$A = \frac{1}{J}\left[\frac{0}{2a_0\left(1+\gamma_1\bar{x}\right)} - \frac{2}{3}a_0^3\left(1+\gamma_1\bar{x}\right)\right] = -\left(1+3\gamma_1\bar{x}\right) \quad (25)$$

or

$$A_V = -\left(1+3\gamma_1\bar{x}\right) \quad (26)$$

Taking into account the resistance of the inhomogeneous viscous elastic base, the equation of motion for a shaft of variable cross-section can be written as follows:

$$E_t^+ J_0 \frac{\partial^2}{\partial x^2}\left[A\left(x\right)\frac{\partial^2 f_1}{\partial x^2}\right] + m_1\left(x\right) f_1 - m_2\left(x\right)\frac{\partial^2 f_1}{\partial x^2} + \eta_0 t\left(x\right)\frac{\partial^2 f_1}{\partial t^2} = 0 \quad (27)$$

We write the differential equation for this case as follows:

$$\frac{\partial^2}{\partial x^2}\left[\left(1+3\gamma_1\bar{x}\right)\frac{\partial^2 f_1}{\partial x^2}\right] + \bar{m}_1\left(x\right)W_1 - \bar{m}_2\left(x\right)\frac{\partial^2 f_1}{\partial x^2} + \eta a_0\left(1+\gamma_1\bar{x}\right)\frac{\partial^2 f_1}{\partial t^2} = 0 \quad (28)$$

$$\left(1+3\gamma_1\bar{x}\right)\frac{\partial^4 f_1}{\partial x^4} + 3\gamma_1 l^{-1}\frac{\partial^3 f_1}{\partial x^3} + \bar{m}_1\left(x\right) f_1 + \bar{m}_2\left(x\right)\frac{\partial^2 f_1}{\partial t^2} + \eta a_0\left(1+\gamma_1\bar{x}\right)\frac{\partial^2 f_1}{\partial t^2} = 0 \quad (29)$$

Equation (29) is a complex partial differential equation. To solve it, we apply the Bubnov Galerkin and separation of variables methods. In the first stage, we look for the deflection (f_1) as follows:

$$f_1 = \vartheta_1\left(x\right)e^{i\omega_1 t} \quad (30)$$

Substituting Equation (24) into the motion Equation (23), we obtain:

$$\left(1+3\gamma_1\bar{x}\right)\frac{d^4\vartheta_1}{dx^4} + 3\varepsilon l^{-1}\frac{d^3\vartheta_1}{dx^3} + \bar{m}_1\left(x\right)\vartheta_1 - \omega_1^2\left[\bar{m}_2\left(x\right) + \eta_0 t\left(1+\gamma_1\bar{x}\right)\right]\vartheta_1 = 0 \quad (31)$$

We look for the deflection function in the following form:

$$\vartheta_1\left(x\right) = \sum_{i=1}^{n} a_i \psi_i\left(x\right) \quad (32)$$

where, the a_i is values are constants that are not known in advance, and each $\psi_i\left(x\right)$ must satisfy the given boundary conditions. Taking Equation (32) into account with (29), the error can be written as follows:

$$\lambda\left(x\right) = \sum_{i=1}^{n} a_i\left[\left(1+3\gamma_1\bar{x}\right)\frac{d^4\psi_i}{dx^4} + 3\gamma_1 l^{-1}\frac{d^3\psi_i}{dx^3} + \bar{m}_i\left(x\right) - \omega_1^2\left(\bar{m}_2\left(x\right) + \eta_0 h\left(1+\gamma_1\bar{x}\right)\right)\right]\psi_i \neq 0 \quad (33)$$

The orthogonality for the error function satisfies the following condition:

$$\int_0^l \lambda\left(x\right)\psi_k dx = 0, \ k = 1, 2, \ldots \quad (34)$$

We obtain the following equation from the first approximation:

$$\int_0^l \left[\left(1+\gamma_1\bar{x}\right)\frac{d^4\psi_1}{dx^4} + 3\gamma l^{-1}\frac{d^3\psi_1}{dx^3} + \bar{m}_1(x)\psi_1 - \right. \\ \left. -\omega_1^2\left[\bar{m}_2(x) + \eta_0 b_0\left(1+\gamma_1\bar{x}\right)\psi_1\right]\psi_1(x) = 0 \right. \tag{35}$$

From this expression, we determine the value of the frequency (ω_1^2):

$$\omega_1^2 = \frac{\int_0^l \left[\left(1+\gamma_1\bar{x}\right)\frac{d^4\psi_1}{dx^4} + 3\gamma l^{-1}\frac{d^3\psi_1}{dx^3} + \bar{m}_1(x)\psi_1 \right]\psi_1(x)dx}{\int_0^l \left[\bar{m}_2(x) + \eta_0 a_0\left(1+\gamma_1\bar{x}\right)\right]\psi_1^2(x)dx} \tag{36}$$

To carry out the calculation, it is necessary to assign specific values to the functions $\psi_1(x)$, $m_1(x)$ and $m_2(x)$. We perform the calculation for a shaft with ends supported by hinges. In this case, can be taken as $\psi_1(x) = \sin\frac{n_1\pi}{l}x$. Let us consider the characteristics of the non-homogeneous foundation and the case where the shaft's properties vary linearly along its length:

$$\bar{m}_1(x) = \left(1+\nu\bar{x}\right)F_1 \\ \bar{m}_2(x) = \left(1+\nu\bar{x}\right)F_2 \tag{37}$$

Equation (37) and taking into account $\psi_1(x) = \sin\frac{n_1\pi}{l}x$, (36) is written as follows:

$$\omega^2 = \frac{\int_0^l \left[\left(\frac{n_1\pi}{l}\right)^4 \left(1+3\gamma_1\bar{x}\right)\sin\frac{n_1\pi}{l}x - 3\gamma_1\left(\frac{n_1\pi}{l}\right)^3\frac{1}{l^{-3}}\cos\frac{n_1\pi}{l}x \right]\sin\frac{n_1\pi}{l}xdx}{\int_0^l \left[R_2\left(1+\varphi_1\bar{x}\right) + \eta_0 a_0\left(1+\gamma_1\bar{x}\right)\right]\sin^2\frac{n_1\pi}{l}xdx} + \\ + \frac{\int_0^l \left[\bar{R}_1\left(1+\varphi\bar{x}\right)\sin\frac{n_1\pi}{l}x\right]\sin\frac{n_1\pi}{l}xdx}{\int_0^l \left[\bar{R}_2\left(1+\varphi\bar{x}\right) + \eta_0 a_0\left(1+\gamma_1\bar{x}\right)\right]\sin^2\frac{n_1\pi}{l}xdx} \tag{38}$$

By performing some simple transformations in Equation (38), the following expression can be obtained:

$$\omega^2 = \frac{\left(\frac{n_1\pi}{l}\right)^4\left(1+1.5\gamma_1\right) + \bar{R}_1\left(1+0.5\varphi\right)}{\bar{R}_2\left(1+0.5\lambda\right) + \eta_0 a_0\left(1+0.5\gamma_1\right)} \tag{39}$$

In expression (39), the following special cases can be considered:

1) The resistance of the inhomogeneous viscoelastic base is not taken into account:

$$\omega_1^2 = \frac{\left(\frac{n_1\pi}{l}\right)^4\left(1+1.5\gamma_1\right)}{\eta_0 a_0\left(1+0.5\gamma_1\right)} = \left(\frac{n_1\pi}{l}\right)^4\left(\eta_0 a_0\right)^{-1}\frac{1+1.5\gamma_1}{1+0.5\gamma_1} \tag{40}$$

2) The beam has a constant thickness along its length and the resistance of the inhomogeneous viscoelastic base is ignored:

$$\omega_1^2 = \left(\frac{n_1\pi}{l}\right)^4 \times \frac{1}{\eta_0 a_0} \tag{41}$$

3) The beam has a constant cross-section along its length and the resistance of the inhomogeneous viscoelastic base is taken into account.

$$\omega_1^2 = \frac{\left(\frac{n_1\pi}{l}\right)^4 + \bar{R}_1\left(1+0.5\varphi\right)}{\bar{R}_2\left(1+0.5\varphi\right) + \eta_0} \tag{42}$$

4) The beam is located on a Winkler basis characterized by a constant along its length:

$$\omega_V^2 = \frac{\left(\frac{n\pi}{l}\right)^4\left(1+1.5\gamma_1\right) + \bar{R}_1\left(1+0.5\varphi\right)}{\eta_0 a_0\left(1+0.5\gamma_1\right)} \tag{43}$$

5) A beam of constant thickness along its length rests on a Winkler base:

$$\omega_V^2 = \frac{\left(\frac{n_1\pi}{l}\right)^4 + \bar{R}_1}{\eta_0 a} \tag{44}$$

To carry out the calculation in the general (complex) case, let us rewrite Equation (38) as follows:

$$\omega_1^2 = \frac{\left(\frac{n_1\pi}{l}\right)^4 + \bar{R}_1\frac{\left(1+0.5\varphi\right)}{1+1.5\gamma_1}}{\bar{R}_2\frac{1+0.5\varphi}{1+1.5\gamma_1} + \eta_0 a_0\frac{\left(1+0.5\gamma_1\right)}{1+1.5\gamma_1}} \tag{45}$$

In equation (45), let us adopt the following substitution:

$$q_1 = \frac{1+0.5\varphi}{1+1.5\gamma_1},\ q_2 = \frac{1+0.5\gamma_1}{1+1.5\gamma_1} \tag{46}$$

Taking expressions (46) into account, equation (45) transforms into the following form:

$$\omega^2 = \frac{\left(\frac{n_1\pi}{l}\right)^4 + q_1\bar{R}_1}{q_1\bar{R}_2 + \eta_0 q_2 a_0} \tag{47}$$

where, the dependence of the parameters q_1 and q_2 on γ_1 and φ is shown through tables and correlation curves. As seen from Figures 2 and 3, as the value of the parameter characterizing the non-homogeneity of the foundation increases, the numerical values of the parameters representing the beams' height and its dimensionless frequency decrease.

The numerical calculations and graphical results show that as the inhomogeneity parameter of the base (α or β) increases, the effective elastic response of the system is distributed unevenly. This leads to an uneven distribution of stiffness along the beam. As a result, the overall dynamic stiffness of the structure decreases. The decrease in stiffness leads to a decrease in the natural vibration frequencies. Mechanical interpretation of the effect of parameters. The conducted study shows that an increase in the parameters of a non-uniform viscoelastic base leads to a decrease in the natural frequencies of the system.

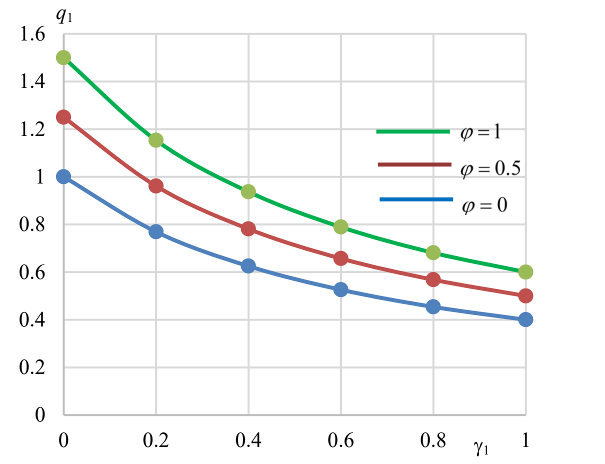

Figure 2. Dependence between the height variation of a beam and the parameters characterizing the non-homogeneity of the foundation

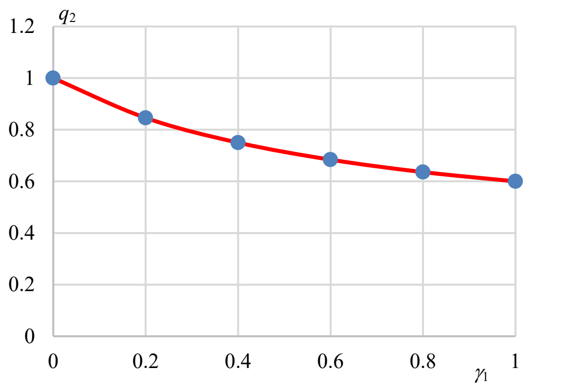

Figure 3. Dependence between the parameters characterizing the height of the beam

This phenomenon is explained from a physical point of view as follows: as the non-uniformity of the base increases, the stiffness distribution in its different parts becomes uneven, and as a result, the effective bearing stiffness of the system decreases. This leads to a softer behavior of the beam and a decrease in the oscillation frequencies.

On the other hand, the analysis conducted regarding the change in the cross section shows that an increase in the stiffness of the beam cross section (i.e. an increase in the bending stiffness EI) increases the overall stiffness of the system, which in turn leads to an increase in the natural frequencies. This result is fully consistent with the classical shaft theory. The difference in the tensile and compressive moduli of the material creates an asymmetric deformation behavior. This phenomenon is especially pronounced in large-amplitude oscillations and affects the change in the effective stiffness.

The obtained results show that the Winkler model, which only responds normally, does not fully reflect the real behavior of the system in some cases. In the Pasternak model, since the shear layer is taken into account, there is an interaction between neighboring points of the foundation. This gives more realistic results, especially in non-uniform foundations.

Comparison of numerical results shows that:
• In the Winkler model, the frequencies are relatively underestimated,
• In the Pasternak model, the frequencies are higher due to the additional stiffness component.

This difference clearly demonstrates the importance of model selection in engineering reports. The variation trends observed in the graphs are consistent with analytical expressions. For example:
• A decrease is observed in the dimensionless frequency expression with an increase in the inhomogeneity parameters,
• The frequency function increases monotonically as the cut-off change function increases.

This consistency confirms the adequacy of the applied Bubnov-Galyorkin method and increases the reliability of the obtained solution.

4. RESULTS AND DISCUSSION

This section presents numerical results obtained as a result of the analysis of the free oscillation motion of a beam made of a material with a variable cross-section and different modulus on a non-uniform Pasternak-type viscoelastic base, and their physical and engineering significance is discussed in detail. The main objective of the study is to determine the effect of the main parameters affecting the dynamic behavior of the beam - the difference in the elastic moduli of the material, the law of variation of the cross-section, as well as the non-uniformity of the mechanical properties of the base - on the oscillation frequencies. The variation of the mechanical properties of a non-uniform viscoelastic base along the beam is described by the function (10) ($\bar{m}_1(x) = (1+\varphi\bar{x})R_1$). The dependence of the non-uniformity parameter of the base on the dimensionless frequency is given in Figure 4.

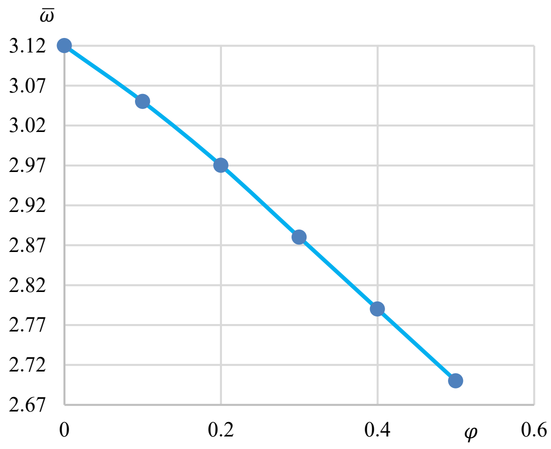

Figure 4. Variation of the inhomogeneity parameter of the base as a function of dimensionless frequency

As can be seen from the figure, the dimensionless frequency decreases monotonically as the inhomogeneity parameter increases. This result is due to the change in the effective stiffness of the system. The uneven distribution of the mechanical properties of the base along the beam leads to a decrease in the elastic resistance in some areas. As a result, the elastic support provided by the base of the beam weakens and the overall effective stiffness of the system decreases. The decrease in the effective stiffness, in turn, leads to a decrease in the natural vibration frequency of the beam.

The change in the cross-section of the beam along the length is assumed by the following function:

$$I(x) = (1+\gamma\bar{x})I_0 \tag{48}$$

where, γ is the variation parameter of the shaft. The dependence of the cross-sectional area variation parameter of a shaft on the dimensionless frequency is shown graphically in Figure 5.

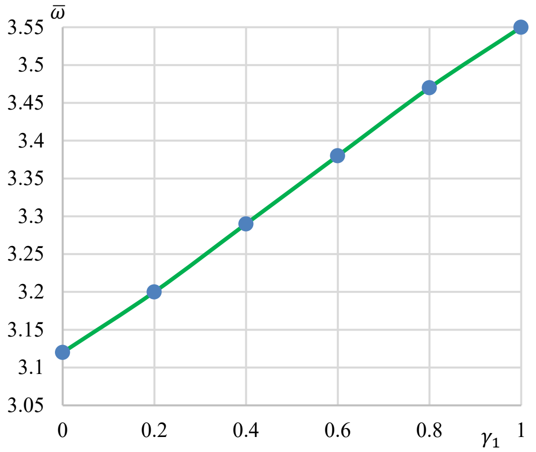

Figure 5. Dependence of the beam cross-section variation parameter on dimensionless frequency

It can be seen from the figure that as the cross-section variation parameter increases, the dimensionless frequency also increases. This result is related to the change in the bending stiffness of the shaft. An increase in the moment of inertia increases the bending resistance of the beam and increases the overall stiffness of the system. As a result, the system exhibits a stiffer behavior and its natural frequency increases. The difference in the tensile and compressive moduli of the material is expressed by Equation (1).

Table 1. Variation of the ratio of elastic moduli as a function of dimensionless frequency

Module ratio, $\mu = \frac{E_h^+}{E_h^-}$	Dimensionless frequency, $\bar{\omega}$
1.0	3.12
1.2	3.18
1.4	3.23
1.6	3.28
1.8	3.32
2.0	3.36

Based on the calculation results, the dependence of the ratio of elastic moduli on the dimensionless frequency is given in Table 1. This is due to the increased stiffness of a beam of varying cross-section in the tension zone. As the modulus of elasticity of the material in tension increases, the elastic resistance in that zone of the beam increases, which increases the overall stiffness of the system. Materials with such mechanical properties are used in many modern engineering structures. The values of dimensionless frequency obtained based on the comparative analysis of the Winkler and Pasternak models are given in Table 1.

Table 2. Reported values of dimensionless frequency according to models

Model	Dimensionless frequency, $\bar{\omega}$
Winkler model	3.18
Pasternak's model	3.25

In the Winkler model, the elastic base is considered as a system of elastic springs operating independently of each other. Although the simplicity of the model has led to its wide application, it does not fully reflect the behavior of the elastic medium in many real engineering systems. In the Pasternak model, the behavior of the elastic base is described more realistically. The results presented in the table show that the adoption of different base models has a significant impact on the value of the natural oscillation frequencies of the system. The dimensionless frequency value obtained in the Winkler model is larger than in other models. The main reason for this is that the model is simplified. Since the interaction between different points of the base is not taken into account in this model, the effective stiffness of the system is estimated somewhat higher, and as a result, the natural frequency is also relatively large.

In the Pasternak model, the behavior of the elastic medium is described more realistically. Here, the deformation process is more complex and the displacement of the beam depends not only on the local elastic response, but also on the interaction of neighboring areas. As a result, the effective stiffness of the system is slightly reduced compared to the Winkler model, and accordingly, the natural vibration frequency also takes a smaller value. The obtained results show that the model of a variable cross-section beam located on a non-uniform Pasternak-type viscoelastic base allows for a more accurate assessment of the vibration characteristics of real engineering systems.

5. CONCLUSIONS

The results of the study show that the free vibrational motion of a beam on a non-uniform viscoelastic base reveals a number of important dynamic behavior features. The results obtained are characterized by the following main points:

The effect of the non-uniformity of the base: Calculations have shown that as the non-uniformity parameter of the base increases, the total effective stiffness of the beam decreases, and as a result, the dimensionless natural frequency ($\bar{\omega}$) also decreases. This is explained from a physical point of view as follows: since the mechanical properties of the base are distributed unevenly along the length, some parts of the beam have weaker support and the system behaves more elastically.

The graphical results show that an increase in the parameter α from 0 to 0.5 leads to a decrease in the dimensionless frequency by about 15-20%. Effect of the cross-sectional variation parameter: When the cross-sectional area of a beam changes along its length ($t(x)=t_0(1+\gamma x)$), the moment of inertia and elasticity distribution become non-uniform. The analysis shows that as the parameter γ increases, the stiffness of the beam increases locally, but the natural frequency of the system decreases overall, together with the inhomogeneity of the foundation. This shows the importance of considering the interaction between the height of the beam and the foundation parameters from an engineering perspective.

Effect of the ratio of the tensile and compressive elastic moduli: The non-symmetric elastic properties of the material ($\mu=\dfrac{E_h^+}{E_h^-}$) have a significant effect on the dynamic stiffness of the beam. As the ratio increases, the stiffness increases more in the tension zone of the beam, while the change in the compression zone is limited. As a result, there is a different effect on the natural frequencies. The analysis shows that when the ratio of the elastic modulus increases from 1 to 2, a 10-12% change in the dimensionless natural frequency is observed.

Comparative calculations show that the Pasternak model more realistically describes the inhomogeneity and damping effect of the foundation, and as a result, the values of natural frequencies and deformation modes are more accurate for the engineer.

Engineering and practical significance: The analysis performed can be applied in the following engineering facility reports:

- In the design of elastically supported beam systems in bridge and pier structures,
- In dynamic analyses of platforms,
- In reports of vibration motion of shafts of drilling and rotating mechanical systems,
- In evaluation of dynamic behavior of composite elements with variable cross-section and different modules.

The proposed model provides more accurate results than the classical Euler–Bernoulli and Timoshenko beam models because it takes into account the heterogeneity of real systems and the internal interaction of the foundation. It is planned to consider nonlinear effects, large deformations, and temperature effects in future research.

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